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Global Stability and Dynamics of Strongly Nonlinear Systems Using Koopman Operator Theory

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14. ABSTRACT
A spectral operator theoretic perspective was taken for modeling nonlinear dynamics subjected to external inputs, due to prescribed forcing or an environment. Under this Director's Research Initiative (DRI), we focused on spectral properties (eigenfunctions, eigenvalues) of the Koopman operator (i.e., an infinite dimensional linear operator that can describe the full dynamics of a nonlinear system). We showed how external forcing could be treated in a Koopman decomposition-based framework. Furthermore, during the course of our study using Koopman decompositions of periodically excited Hopf bifurcation systems, we characterized a new attractor that had never been described in the literature or texts. We called this attractor the "quasi-periodic intermittency" attractor and showed that it may be a fundamental building block of a variety of multi-scale dynamics systems. Perhaps the most far reaching impact of this DRI will be a contribution that was not planned in the original proposal. This contribution has to do with the generalization of Koopman decompositions using a fractional calculus perspective on complexity. By using a combination of Koopman operator theory and fractional calculus, we showed that our generalized spectral decompositions are better suited for complex systems influenced by an external environment in which long-term memory is introduced.

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1. Introduction

Complex, nonlinear dynamical systems are pervasive across many Army relevant scientific disciplines. Typically, our engineering objectives are to control the dynamics (e.g., fluids, polymers), understand underlying phenomena (e.g., neuroscience models, turbulence, networks), and/or synthetically mimic natural phenomena (e.g., swarm dynamics, self-assemblies, active matter/fluids). However, our inability to achieve these objectives for a variety of high-dimensional dynamical systems is due to a lack of mathematical tools to describe low-dimensional (simple) features that may underpin such complex systems.

In this 3-year effort, we focused on one perspective for extracting simplicity in the form of low-dimensional features that otherwise would be hidden by complexity (high dimensionality and multi-scale dynamics). This perspective is based on the theoretical constructs developed in the 1930s by mathematician Bernard Koopman,[1] who showed that finite dimensional nonlinear dynamics can be fully described by an infinite dimensional linear operator. This operator is referred to as the Koopman operator and the resulting projections as Koopman decompositions.[2] Furthermore, as described in Budisic and Mezic's research,[2] the Koopman operator approach entails tracking dynamics of observables (i.e., measureable quantities), rather than dynamics of states. This perspective makes Koopman operator approaches ideal for dealing with ill-described systems in only measurement data available rather than access to model equations.

The contributions of this effort focused on the broad area of dynamical systems subjected to external forcing. We considered 2 scenarios: the first in which an external forcing is applied to a nonlinear Hopf bifurcation, and the second in which the external environment induces long-time memory in the nonlinear system. Details of our work in the first scenario are described by Glaz et al.,[3] while details of the second are provided in the research of Svenkeson et al.[4] Section 2 is Accomplishments and Section 3 is the Conclusion.

2. Accomplishments

2.1 Prescribed External Forcing

To study the impact of external forcing, we studied a canonical Hopf bifurcation system with external forcing. Fluid dynamics induced by periodically forced flow around a cylinder was analyzed computationally for the case when the forcing frequency is much lower than the von Karman vortex shedding frequency (i.e., Hopf bifurcation frequency) corresponding to the constant flow velocity condition. Beginning from the generic dynamical system,

$$\dot{\boldsymbol{z}} = \boldsymbol{F}(\boldsymbol{z}), \tag{1}$$

we introduced a prescribed forcing **U**(t) by oscillating the cylinder in the streamwise direction.[3] This created a simple canonical situation for studying the forced bifurcation system under potentially high-dimensional conditions. The forced system is then

$$\dot{\boldsymbol{z}} = \boldsymbol{F}(\boldsymbol{z}) + \boldsymbol{U}(t). \tag{2}$$

However, in the dynamics of observables perspective we are interested in tracking a measurable quantity **g** that may be a nonlinear function of state **z** (i.e., **g(z)**). In operator form, we showed that the forced system in observable space is described by a bi-linear equation,

$$\dot{\boldsymbol{g}} = \boldsymbol{L}\mathbf{g} + (\boldsymbol{U} \cdot \boldsymbol{\nabla})\mathbf{g}, \tag{3}$$

where $\boldsymbol{L}$ is a linear operator that describes the dynamics due to the unforced system, Eq. 1, while the forcing appears as a bi-linear term involving $\boldsymbol{U}$ and **g.**

By using the Koopman mode decomposition approach, we found a new normal form equation that extends the classical Hopf bifurcation normal form by a time-dependent term for Reynolds numbers close to the Hopf bifurcation value. This was done by projecting onto the first Koopman mode associated with the Hopf bifurcation and obtaining a reduced order set of equations.[3] The resulting 2-dimensional (2-D) normal form equation was

$$\dot{\eta} = \lambda\eta + \beta\eta^2\bar{\eta} - iQ(\zeta - \bar{\zeta})\eta, \tag{4}$$

where η is a complex valued variable representing the observable, ζ is a complex valued variable representing the simple harmonic prescribed forcing term, and the remaining terms parameterize the dynamical system and the forcing amplitude. As was shown by Glaz et al.[3] using normal form theory, Eq. 4 is only necessary when the forcing frequency is much less than the Hopf bifurcation frequency. Otherwise, nonlinear interactions between the observable and forcing do not result in first-order effects. The validity of the simple 2-D model in Eq. 4 is shown in Fig. 1. In Fig. 1, there is rich spectral content even though there are only 2 underlying frequencies, the Hopf bifurcation frequency (26 Hz) and the prescribed forcing frequency (0.5 Hz). However, the muli-scale nonlinear interaction between the 2 scales is captured by the Koopman operator-based approach. Furthermore, the

Koopman operator approach correctly reproduces the underlying simplicity of the problem in that only one mode is relevant. However, from simply looking at the spectrum of the forced system, one may incorrectly conclude that multiple modes have been introduced by forcing this nonlinear system.

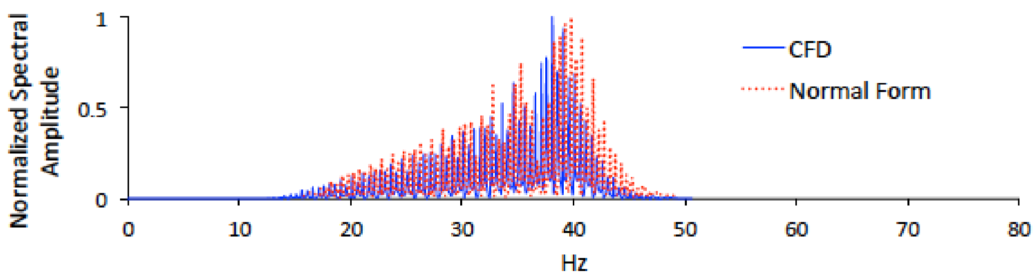

Fig. 1 Spectrum of nonlinear observable; computational fluid dynamics simulation (blue) are for the full-order computational fluid dynamics simulations of the forced system, while normal form (red) corresponds to Eq. 4

Furthermore, we found that the dynamics of the flow in this regime are characterized by alternating instances of quiescent and strong oscillatory behavior, and that this pattern persists indefinitely. We establish the theoretical underpinnings of this phenomenon, that we name quasi-periodic intermittency, using the new normal form model in Eq. 4 and showed that the dynamics are caused by the tendency of the flow to oscillate between the unstable fixed point and the stable limit cycle of the unforced flow. The quasi-periodic intermittency phenomena is also characterized by positive Finite-Time Lyapunov Exponents that, over a long period of time, asymptotically approach zero.[3] The quasi-periodic intermittency attractor is shown in Fig. 2 for the same forced Hopf bifurcation case associated with Fig. 1. Because of the pervasiveness of the underlying ingredients of quasi-periodic intermittency, the attractor is a potentially fundamental building block that may underpin a variety of multi-scale dynamical systems.

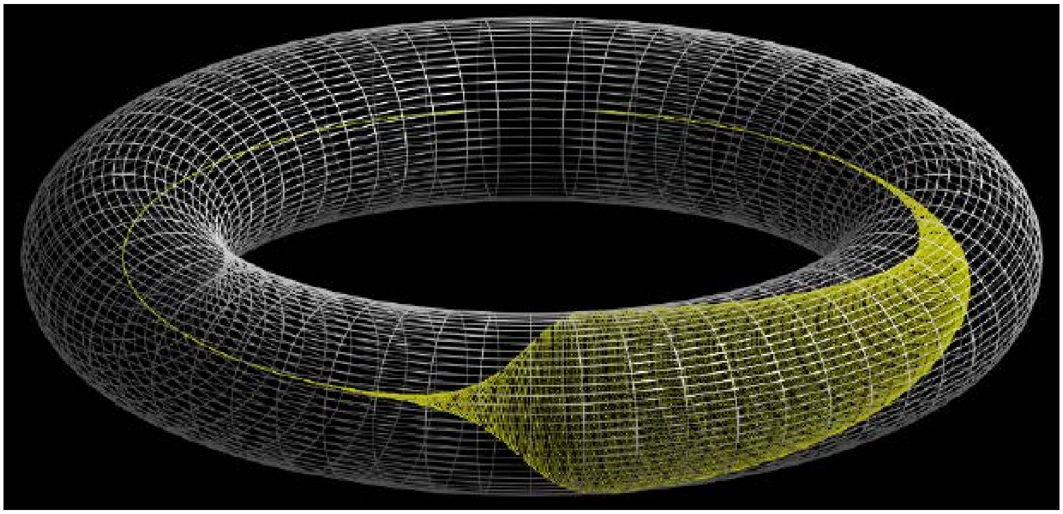

Fig. 2 The quasi-periodic intermittency attractor (yellow) plotted on the torus. The toroidal grid (grey) is shown for reference. Rotation around the large diameter of the torus corresponds to the radial growth/decay that occurs over the slow forcing frequency, while rotation around the smaller diameter (i.e., phase velocity as the attractor spins around cross sections of the torus) corresponds to the faster natural frequency of the system.

2.2 Spectral Decompositions of Nonlinear Systems with Memory

Following the extension of Koopman operator theory to systems with memory in time made during Year 1, during Year 2 we worked toward developing a test capable of determining whether an ill-described (black-box) system is better represented by ordinary differential equations or fractional differential equations. The physical interpretation of the fractional order system is that of a surrounding environment acting on a system.[5] However, the theory in Stanislavsky[5] was limited to that of a linear fractional order oscillator. An important contribution of our work in[4] is that the analogy of a fractional order oscillator to a physical system surrounded by an environment can be extended to nonlinear systems by utilizing the Koopman operator perspective as a starting point.

Traditional modeling approaches assume the memoryless system

$$\frac{d}{dt}x = F(x). \tag{5}$$

However, in some instances (e.g., when the system of interest is coupled to a complex environment) it may be advantageous to assume a system with long-term memory in time, where memory effects are represented by fractional order differential operators,

$$\frac{d^\alpha}{dt^\alpha}x = F(x). \tag{6}$$

The Koopman spectral decomposition of the memoryless system has the form

$$g(x(t)) = \sum_{k=1}^{\infty} V_k e^{\lambda_k t} \tag{7}$$

with exponential expansion functions. The spectral decomposition of a system with memory has the form

$$g(x(t)) = \sum_{k=1}^{\infty} V_k E_\alpha(\lambda_k t^\alpha) \tag{8}$$

with Mittag-Leffler expansion functions; we previously derived Eq. 8 through an extension of the Koopman operator theory. Given a data set originating from observations of a black-box system, we now have 2 tools in Eqs. 7 and 8 that can potentially be used to construct 2 different models of the system. Methods exist for approximating the eigenvalues λ_k and modes V_k of the memoryless spectral decomposition, such as generalized Laplace analysis and dynamic mode decomposition.

The analysis leads to the decomposition of a nonlinear system with memory into modes whose temporal behavior is anomalous and lacks a characteristic scale. On average, the time evolution of a mode follows a Mittag-Leffler function, and the system can be described using the fractional calculus. When analyzing data from an ill-defined (black-box) system, the spectral decomposition in terms of Mittag-Leffler functions that we proposed may uncover inherent memory effects through identification of a small set of dynamically relevant structures that would otherwise be obscured by conventional spectral methods. Consequently, the theoretical concepts we present may be useful for developing more general methods for numerical modeling that are able to determine whether observables of a dynamical system are better represented by memoryless operators, or operators with long-term memory in time, when model details are unknown.

3. Conclusion

While we made significant progress in accounting for external forcing within a Koopman operator-based framework, much work remains to develop a control-oriented methodology. The addition of a prescribed external force can be considered an open loop control example. In this, we showed that the forcing in a Koopman-mode-based reduced order model must appear as a bi-linear term. Future work should address the inclusion of feedback control capabilities using this term.

Furthermore, a key element of an approach that is fully capable of dealing with nonlinear systems is the ability to detect and include new modes that may be introduced by forcing (or far from equilibrium dynamics).

No methods exist yet for approximating the eigenvalues and modes in the spectral decomposition for systems with memory involving Mittag-Leffler functions. However, in principle we can compare the spectral properties of both types of decompositions, memoryless and with memory. The caveat is that we still need to develop numerical methods for approximating the eigenvalues and modes in Eq. 8 to be able to test a data set for memory effects. One direction that could be pursued is to extend the nonlinear optimization methods to be compatible with Mittag-Leffler functions rather than the exponential time dependence.

Properly extracting the modes derived from a mechanistic process, though seemingly without pattern or chaotic, is a key contribution of this work. While the introduction of fractional calculus to obtain a more general spectral decomposition approach may be more effective in properly extracting coherency from complex systems, we have yet to investigate systems with stochastic features. In many data driven system, randomness will be present in the measurements. Future work should address the effects of stochastic/random noise in the system such that spectral decompositions identify coherent features (modes) and appropriate interactions/effects of the noise. Once this is done, one could develop control-oriented reduced-order models that focus on the deterministic dynamics, the random effects, or both.

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